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ON A MATHEMATICAL PROBLEM OF THE THEORY OF ELASTIC STABILITY

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V.S. KOLESOV, Iu.S. KOLESOV, A.N. KULIKOV and I.I. FEDIK
(Moscow, Iaroslavl')

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There is considered the problem of bifurcation of the self-oscillations from the equilibrium state for nonlinear differential equations of hyperbolic type. A general qualitative analysis is performed and formulas are presented to compute the parameters of periodic solutions.

Loss of stability occurs in many nonconservative problems of elastic stability theory (see [1, 3], for instance) because the pair of complex-conjugate eigenvalues belonging to the stability spectrum of the linearized problem intersects the imaginary axis under changes in some parameters and goes over into the right half-plane of the complex plane. In this case, the analysis of the appropriate mechanical problem in a nonlinear formulation will permit clarification of the soft or hard nature of loss of stability. One of the best known of such problems is nonlinear "panel flutter" at supersonic speeds, which is simulated by using the law of plane sections [4]. The mathematical methods used in this latter problem reduce to the replacement of the nonlinear partial differential equations by their Galerkin approximations with subsequent application of the method of a small parameter of the harmonic balance method. As has been shown in [5], the results obtained by this means are often far from the correct one .

A strict method of investigating appropriate nonlinear partial differential equations is proposed below. The mathematical apparatus to be used here is a combination of the method of integral manifolds and a method developed in [6]. The method of integral manifolds is not applicable here in pure form. This is related to the specific hyperbolic-type equations for which the properties of the solutions cannot assure the necessary smoothness of the invariant surface [7]. Let us note that the problem being solved below has actually been solved by V. V. Bolotin (see [1], p. 333).

1. Description of the class of differential equations under consideration. Below, E is a real Banach space, R^m is e Euclidean m-space. We consider in E the equation

$$x^{\cdot\cdot} + A(\varepsilon) x^{\cdot} + [B^2 + C(\varepsilon)]x = f(x, x^{\cdot}; \varepsilon, \mu) \tag{1.1}$$

dependent on a numerical parameter ε and a vector parameter $\mu \in R^m$. It is later assumed that

$$|\varepsilon| \leqslant \varepsilon_0, \quad \|\mu\|_{R^m} \leqslant \mu_0 \tag{1.2}$$

where ε_0 and μ_0 are sufficiently small. Let us write the constraints imposed on the coefficients of (1.1). We first examine those which refer to the linear part

$$x^{\cdot\cdot} + A\ (\varepsilon)\ x^{\cdot} + [B^2 + C\ (\varepsilon)]\ x = 0 \tag{1.3}$$

We shall assume that B is a closed, linear, unbounded operator with domain of definition $D\ (B)$ compact in E, and iB, the generating operator of the group of class (C_0) in the complex expansion of E and B has a continuous inverse B^{-1} in E. We shall later assume that $\| C\ (\varepsilon)B^{-1} \|_E \leqslant$ const and the operators $C\ (\varepsilon)B^{-1}$ and $A\ (\varepsilon)$depend analytically on ε in the metric of linear operators acting from E into E. This latter constraint concerning (1.3) is the following. We introduce the quadratic sheaf

$$L\ (\lambda;\ \varepsilon) = \lambda^2 I + \lambda A\ (\varepsilon) + B^2 + C\ (\varepsilon) \tag{1.4}$$

into the considerations, where I is the unit operator. We call the points of its spectrum the stability spectrum by considering that the nature of their arrangement governs the behavior of the solutions of (1.3). We shall consider that two simple eigenvalues $\lambda\ (\varepsilon) = \tau\ (\varepsilon) + i\sigma\ (\varepsilon)$, $\bar{\lambda}\ (\varepsilon) = \tau\ (\varepsilon) - i\sigma\ (\varepsilon)$, which evidently depend analytically on ε and for which we consider the following conditions satisfied

$$\tau(0) = 0, \quad \sigma_0 = \sigma(0) > 0, \quad \tau_0' = \frac{d}{d\varepsilon}\tau(\varepsilon)\,|_{\varepsilon=0} \neq 0$$

belong to it.

With respect to the remaining points of the stability spectrum, we shall assume that they are located in the part of the complex plane extracted by the inequality $\operatorname{Re} \lambda \leqslant -\gamma_0 < 0$.

Let us turn to a description of the constraints imposed on the right side of (1.1). To this end, we introduce the spaces $E\ (B)$ and $E\ (B^2)$ which consist of the elements $x \in D\ (B)$ and $x \in D\ (B^2)$, respectively, and are normalized as follows

$$\| x \|_{E(B)} = \| Bx \|_E, \quad \| x \|_{E(B^2)} = \| B^2 x \|_E$$

Let us assume that the operator $f\,(x,\ y;\ \varepsilon,\ \mu)$ acts from some sphere of sufficiently small radius in the space $E\ (B) \times E\ (B) \times R^{m+1}$ into E and is analytic in a set of variables, where it has an order of smallness higher than the first in the variables x and y at zero. Let us assume that the Frechet derivative of $f\ (x,\ y;\ \varepsilon,\ \mu)$ with respect to y admits of extension to a continuous linear operator $D_y f\ (x,\ y;\ \varepsilon,\ \mu)$ acting from E into E, where this extension continuously depends strongly on ε and μ, and satisfies the Lipschitz condition with the universal constant N_0

$$\begin{gathered} \| D_y f\,(x_1,\ y_1;\ \varepsilon,\ \mu) - D_y f\,(x_2,\ y_2;\ \varepsilon,\ \mu) \|_{E\to E} \leqslant \\ N_0\,[\| x_1 - x_2 \|_{E(B)} + \| y_1 - y_2 \|_{E(B)}] \end{gathered} \tag{1.5}$$

in the variables x and y belonging to some sphere of the space $E\ (B) \times E\ (B)$.

This latter constraint is the following. We shall assume that the operator $f\ (x,\ y;\ \varepsilon,\ \mu)$, for values of the parameters ε and μ satisfying the inequalities (1.2) and for each sufficiently small $\delta > 0$, will allow a space $f_\delta\ (x,\ y;\ \varepsilon,\ \mu)$ with the elements

$$\| x \|_{E(B)} \leqslant \delta, \quad \| y \|_{E(B)} \leqslant \delta \tag{1.6}$$

in all elements of $E\ (B) \times E\ (B)$ such that

$$f_\delta\,(x,\ y;\ \varepsilon,\ \mu) \equiv f\,(x,\ y;\ \varepsilon,\ \mu)$$

for x and y satisfying the inequalities (1.6), that for arbitrary x and y from $E(B) \times E(B)$ the operator $f_\delta(x, y; \varepsilon, \mu)$ possesses all the properties of the operator $f(x, y; \varepsilon, \mu)$ with analyticity replaced by infinite differentiability, that the inequality (1.5) with a certain constant N_δ common to all emements of $E(B) \times E(B)$ is satisfied, and that

$$\| f_\delta(x_1, y_1; \varepsilon, \mu) - f_\delta(x_2, y_2; \varepsilon, \mu) \|_E \leqslant q(\delta)\,[\| x_1 - x_2 \|_{E(B)} + \| y_1 - y_2 \|_{E(B)}]$$

in all elements of the space $E(B) \times E(B)$, where the function $q(\delta)$ decreases monotonically to zero as $\delta \to 0$.

2. Formulation of the problem. As usual, we shall call the function

$$x(t) \in C^2([-T, T], E) \cap C^1([-T, T], E(B)) \cap C([-T, T], E(B^2))$$

which turns (1.1) into an identity for $-T \leqslant t \leqslant T$ and which satisfies the initial conditions $x(0) = x_0 \in E(B^2)$, $x^{\cdot}(0) = x_0^{\cdot} \in E(B)$, the Cauchy solution. Let $S(r)$ denote a sphere of radius $r > 0$ with center at the zero of the space $E(B^2) \times E(B)$. There results from [8] that for any $T > 0$ a sphere $S[r(T)]$ of radius $r(T) > 0$ exisits with initial conditions from which the Cauchy problem for (1.1) is uniquely solvable. Let us examine the question of nonlocal continuability of the solutions of this equation with initial conditions from a certain fixed sphere $S(r_0)$ and the question of the behavior of those solutions as $t \to \infty$, which belong to this sphere for all $t \geqslant 0$. Such solutions are most important for applications since solutions which are sufficiently great in the norm are not generally physically realizable: disruption of the system occurs for appropriate values of t.

3. Fundamental results. Let us first introduce a number of notations and concepts. Henceforth $e_0 = e_1 + ie_2$ is an eigenelement of the operator sheaf $L(\lambda; 0)$ corresponding to the eigenvalue $i\sigma_0$, and $h_0 = h_1 + ih_2$ is a linear functional which is the eigenelement of the adjoint operator sheaf

$$L^*(\lambda;\ 0) = \lambda^2 I - \lambda A^*(0) + B^{*2} + C^*(0)$$

corresponding to the same eigenvalue. As is known, it can be considered that

$$(h_j,\ e_k) = \delta_{jk} \quad (j\ k = 1, 2) \tag{3.1}$$

where δ_{jk} is the Kronecker delta. Here $(a,\ b)$ in (3.1) denotes the value of the functional a at the element b. Using e_0 and h_0 we introduce the functions

$$E_1(t) = e_1 \cos \sigma_0 t - e_2 \sin \sigma_0 t, \quad E_2(t) = e_1 \sin \sigma_0 t + e_2 \cos \sigma_0 t$$

$$H_1(t) = h_1 \cos \sigma_0 t - h_2 \sin \sigma_0 t, \quad H_2(t) = h_1 \sin \sigma_0 t + h_2 \cos \sigma_0 t$$

into the considerations, which are needed for examination of the question of the existence of $2\pi/\sigma_0$-periodic solutions for the inhomogeneous equation

$$x^{\cdot\cdot} + A(0)\, x^{\cdot} + [B^2 + C(0)]x = f(t) \quad (f(t + 2\pi/\sigma_0) \equiv f(t)) \tag{3.2}$$

with a sufficiently smooth right side. Namely,(3.2) has periodic solutions if and only if

$$m_k\,[f(t)] = \frac{\sigma_0}{2\pi} \int_0^{2\pi/\sigma_0} (H_k(t),\, f(t))\,dt = 0 \quad (k = 1,\, 2) \tag{3.3}$$

Furthermore, let us define two classes of functions with special properties.

The first class W consists of real scalar functions $w(\xi,\mu)$. By definition, $w(\xi, \mu) \in W$ if

1°. The function $w(\xi, \mu)$ is analytic in the set of variables for

$$|\xi| \leqslant \xi_0, \quad \|\mu\|_{R^m} \leqslant \mu_0 \tag{3.4}$$

2°. For ξ and μ satisfying the inequalities (3.4), the conditions

$$w(\xi, \mu) \equiv w(-\xi, \mu), \quad w(0, \mu) \equiv 0$$

are satisfied.

The second class X_τ consists of functions $x(\tau; \xi, \mu)$, periodic in τ with period $2\pi / \sigma_0$, with values in $E(B^2)$ which additionally satisfy the conditions

1°. The functions $x(\tau; \xi, \mu)$ are analytic in the set of variables in the metric of the space $E(B^2)$ for all τ and values ξ and μ satisfying the inequalities (3.4).

2°. For the same ξ and μ the identities

$$m_1\,[x(\tau; \xi, \mu)] \equiv \xi, \quad m_2\,[x(\tau; \xi, \mu)] \equiv 0$$
$$x(\tau; 0, \mu) \equiv 0, \quad \frac{\partial}{\partial\xi} x(\tau; \xi, \mu)|_{\xi=0} \equiv E_1(\tau)$$

are satisfied.

Now, let us examine the differential equation

$$\frac{d^2x}{d\tau^2} + (1+c)\,A(\varepsilon)\,\frac{dx}{d\tau} + (1+c)^2\,[B^2 + C(\varepsilon)]\,x = (1+c)^2 f\left(x,\, \frac{1}{1+c}\,\frac{dx}{d\tau};\, \varepsilon,\, \mu\right) \tag{3.5}$$

which is obtained from (1.1) by using the substitution $t = (1+c)\,\tau$, where $|c| < 1$. We shall consider (3.5) as an equation in

$$c = c(\xi, \mu) \in W, \quad \varepsilon = \psi(\xi, \mu) \in W \tag{3.6}$$
$$x = x(\tau; \xi, \mu) \in X_\tau$$

Theorem 1. There exists a single set of functions (3.6) turning (3.5) into an identity.

From this proposition there results that to each solution $\xi = \xi(\varepsilon, \mu)$ of the scalar equation

$$\varepsilon = \psi(\xi, \mu) \tag{3.7}$$

there corresponds a periodic solution

$$x(t; \varepsilon, \mu) = x\left(\frac{t}{1 + c(\xi(\varepsilon, \mu)\ \mu)};\, \xi(\varepsilon, \mu),\, \mu\right) \tag{3.8}$$

of (1.1). The natural question arises: are all geometrically distinct periodic solutions of (1.1) exhausted in this manner for each t belonging to some sphere $S(r_0)$ of the phase space $E(B^2) \times E(B)$. Let us recall that periodic solutions which are impossible to obtain from each other by time shifts are called geometrically distinct.

We shall assume everywhere below that $\psi(\xi, 0) \not\equiv 0$. Under this condition and for sufficiently small ε_0 and μ_0 in the right sides of the inequalities (1.2), solutions passing through the ends of a sufficiently small segment $[-\xi_0, \xi_0]$ cannot appear for the scalar equation (3.7) for arbitrary change in ε and μ in the appropriate limits. Henceforth, we shall assume that the smallness of the ε_0, μ_0 and ξ_0 necessary for this is conserved.

Theorem 2. There exists an r_0 and a ε_0, μ_0 and ξ_0 dependent on it such that for ε and μ satisfying the inequalities (1.2), the differential equation (1.1) has as many geometrically distinct periodic solutions for each t belonging to $S(r_0)$ as (3.8) has distinct solutions belonging to the interval $(0, \xi_0)$.

Theorem 3. The trajectory of each solution of the differential equation (1.1) asymptotically approaches either the zero equilibrium state or one of its cycles for all $t \geqslant 0$ belonging to $S(r_0)$.

Let us turn to the question of the Liapunov stability of the periodic solutions constructed earlier.

Theorem 4. For given ε_* and μ_* let the scalar equation (3.7) have the simple solution $\xi_* \in (0, \xi_0)$. Then the periodic solution $x(t; \varepsilon_*, \mu_*)$ constructed thereby according to (3.8) is stable (unstable) if

$$\tau_0' \frac{\partial}{\partial \xi} \psi(\xi, \mu) |_{\xi=\xi_*, \mu=\mu_*} > 0 \quad (< 0)$$

There remains to explain the behavior of the solutions when the scalar equation (3.7) has no solutions in the interval $(0, \xi_0)$.

Theorem 5. For given ε_* and μ_* all the solutions of the differential equation (1.1) with initial conditions from the sphere $S(r_0)$ asymptotically approach the zero solution if

$$\tau_0' [\varepsilon_* - \psi(\xi, \mu_*)] < 0 \quad (0 < \xi < \xi_0)$$

and this equation has a solution with initial conditions arbitrarily small with respect to the norm, which leave the sphere $S(r_0)$ with the lapse of time if the inequality mentioned is replaced by its opposite.

4. Algorithmic part. Let

$$\begin{aligned} c(\xi, \mu) &= c_2(\mu)\xi^2 + c_4(\mu)\xi^4 + \ldots \\ \psi(\xi, \mu) &= b_2(\mu)\xi^2 + b_4(\mu)\xi^4 + \ldots \\ x(\tau; \xi, \mu) &= \xi E_1(\tau) + \xi^2 x_2(\tau; \mu) + \ldots \end{aligned} \tag{4.1}$$

be series expansions of the functions (3.6) in powers of ξ. Substituting them into (3.5), expanding the left and right sides of the identities obtained in power series in ξ and equating coefficients of identical powers of ξ, we obtain recursion sequences of linear inhomogeneous differential equations

$$\frac{d^2}{d\tau^2} x_n(\tau;\mu) + A(0)\frac{d}{d\tau} x_n(\tau;\mu) + \tag{4.2}$$
$$[B^2 + C(0)]\, x_n(\tau;\mu) = \varphi_n(\tau;\mu), \quad n = 2k,\ 2k+1$$
$$\varphi_{2k}(\tau;\mu) = \varphi_{2k}(\tau;\, c_2(\mu), \ldots, c_{2k-2}(\mu), b_2(\mu), \ldots, b_{2k-2}(\mu), \mu)$$
$$\varphi_{2k+1}(\tau;\mu) = \varphi_{2k+1}(\tau;\, c_2(\mu), \ldots, c_{2k}(\mu), b_2(\mu), \ldots$$
$$\ldots, b_{2k}(\mu), \mu), \quad k = 1, 2 \ldots$$

Here φ_{2k}, φ_{2k+1} are trigonometric polynomials of the variable τ with numbers of the harmonics not exceeding $2k$ and $2k+1$, respectively. As it turns out

$$m_1[\varphi_{2k}(\tau;\mu)] = m_2[\varphi_{2k}(\tau;\mu)] = 0$$

always. The equalities

$$m_1[\varphi_{2k+1}(\tau;\mu)] = m_2[\varphi_{2k+1}(\tau;\mu)] = 0$$

uniquely define the coefficients of the first two series in (4.1). The coefficients of the last series in (4.1) are hence defined uniquely as trigonometric solutions of (4.2) satisfying the equalities

$$m_k[x_j(\tau;\mu)] = 0 \quad (k = 1, 2; \quad j = 2, 3, \ldots)$$

Often $b_2(0) \neq 0$ in applications. In this case, the scalar equation (3.7) cannot have more than one solution in the sufficiently small interval $(0, \xi_0)$ for sufficiently small ε_0 and μ_0 If such a solution exists, then it is expanded in the series

$$\xi = a_1(\mu)\varepsilon^{1/2} + a_3(\mu)\varepsilon^{3/2} + \ldots$$

of odd powers of $\varepsilon^{1/2}$. It can hence be considered that $\xi \approx a_1(\mu)\varepsilon^{1/2}$. Then it follows from (3.8) that

$$x(t;\varepsilon,\mu) \approx a_1(\mu)\,\varepsilon^{1/2} E_1\left(\frac{t}{1 + c_2(\mu)\, a_1^2(\mu)\,\varepsilon}\right) \tag{4.3}$$

Let us still note that the parameter μ reflects the influence of different nonlinearities in applications. For instance, the geometric and aerodynamic nonlinearitites in the panel flutter problem. If it turns out that $b_2(0) = 0$ for a certain interaction, then usually $b_4(0) \neq 0$. It is clear that in this case also the scalar equation (3.7) can be analyzed completely.

5. Proofs. All the propositions formulated have analogs in [6], in which it has been noted that the appropriate foundations are general in nature and applicable to a broad class of evolutionary equations. In this connection, we indicate only changes due to singularities in the case under consideration.

In combination, the algorithmic part and Theorem 1 are approximately equivalent to the material in [9, 10], where parabolic equations were studied. They were given a foundation by the scheme elucidated in [6], and no new method occurs here. Let us note that this scheme is similar to that used in [9, 10].

Let us proceed to give Theorem 2 a foundation. To do this we introduce the differential equation

$$u^{\cdot} = Q(\varepsilon)u + F_\delta(u;\varepsilon,\mu) \tag{5.1}$$

obtained from (1.1) in which the function $f(x, x^{\cdot}; \varepsilon, \mu)$ is replaced by $f_\delta(x, x^{\cdot}; \varepsilon, \mu)$, by using the substitution

$$u = (x_1, x_2), \quad x_1 = Bx, \; x_2 = x^{\cdot}$$

The Cauchy problem for (5.1) is solvable nonlocally for any initial condition $u \in E(B) \times E(B)$ and the set of its solutions forms a group of nonlinear operators

$$U(t, u; \varepsilon, \mu) = T(t; \varepsilon)u + V(t, u; \varepsilon, \mu) \qquad (5.2)$$

where $T(t; \varepsilon)$ is a group of linear operators of the class (C_0) with generating operator $Q(\varepsilon)$ [11]. The nonlinear member in the right side of (5.2) satisfies all the constraints formulated in [7]. In particular, the operator $V(t, u; \varepsilon, \mu)$, as an operator from $E(B) \times E(B) \times R^{m+1}$ into $E(B) \times E(B)$, is continuous in the set of variables, and it satisfies the Lipschitz condition with a sufficiently small constant in the space variable, and has an order of smallness greater than the first at zero.

Let us note that the stability spectrum of (1.3) agrees with the spectrum of the operator $Q(\varepsilon)$. Let $e_1(\varepsilon) + ie_2(\varepsilon)$ denote the eigenelement of the operator $Q(\varepsilon)$ corresponding to the eigenvalue $\lambda(\varepsilon)$, and let $h_1(\varepsilon) + ih_2(\varepsilon)$ denote a linear functional which is the eigenelement of the adjoint operator $Q^*(\varepsilon)$ and corresponds to the eigenvalue $\bar{\lambda}(\varepsilon)$. They can be selected independently of ε, smoothly and such that $(h_k(\varepsilon), e_j(\varepsilon)) = \delta_{kj}$, where $k, j = 1, 2$. These quantities are necessary for the phase space $E(B) \times E(B)$ of (5.1) to decompose into the direct sum of two subspaces $C_1(\varepsilon)$ and $C_2(\varepsilon)$. The former is the linear envelope of $e_1(\varepsilon)$ and $e_2(\varepsilon)$, while the latter is given by the equalities $(h_k(\varepsilon), u) = 0$, where $k = 1, 2$. We denote their corresponding projectors by $P_1(\varepsilon)$ and $P_2(\varepsilon)$. We denote the set of elements related by the inequality

$$\rho \| P_1(\varepsilon) u \|_\varepsilon \geqslant \| P_2(\varepsilon) u \|_\varepsilon \quad (\rho = \text{const} > 0)$$

by $M(\rho, \varepsilon)$.

Here $\| * \|_\varepsilon$ is some special norm in $E(B) \times E(B)$ which is uniformly equivalent in ε to the ordinary norm which is selected as in [6]. There results from the properties of the nonlinear group (5.2) and the results of [6] that the solutions of (5.1) with the initial conditions $u \in M(\rho, \varepsilon)$ either fall into this set with the lapse of time by decreasing in the norm, or appoach the zero solution exponentially. Rotation (see [6]) occurs in the very same set $M(\rho, \varepsilon)$. These two facts permit reduction of the question of the behavior of solutions of (5.1) to the question of the behavior of iterations of the Poincaré operator

$$\Pi(u; \varepsilon, \mu) = U(t_K(u; \varepsilon, \mu), u; \varepsilon, \mu) \qquad (5.3)$$

defined in the set $K(\rho, \varepsilon)$ consisting of $u \in M(\rho, \varepsilon)$ such that $(h_1(\varepsilon), u) \geqslant 0$, and $(h_2(\varepsilon), u) = 0$. The functional $t_K = t_K(u; \varepsilon, \mu)$ in (5.3) is a continuous solution of the equation

$$\exp \tau(\varepsilon) t_K \sin \sigma(\varepsilon) t_K = (h_2(\varepsilon), V(t_K, u; \varepsilon, \mu))$$
$$t_K(0; \varepsilon, \mu) = 2\pi/\sigma(\varepsilon)$$

Such a solution in the space variable certainly satisfies the Lipschitz condition. The subsequent reasoning differs somewhat from that elucidated in [6], where the monotonicity of the operator (5.3) in the sense of the semi-order generated by the cone $K(\rho, \varepsilon)$, was used. In the case of hyperbolic type equations this fact is not established successfully since it is impossible to extract the principal linear part from the nonlinear operator (5.3).

It follows from the results in [7] that the operator (5.3) has a unique invariant curve $l(\varepsilon, \mu)$ in $K(\rho, \varepsilon)$ with the following properties: it is given by the equation $u = \eta e_1(\varepsilon) + v_0(\eta; \varepsilon, \mu)$, where the parameter $\eta \geqslant 0$; the function $v_0(\eta; \varepsilon, \mu)$, continuous in the set of variables and with values in $C_2(\varepsilon)$, satisfies the Lipschitz condition with respect to η with a constant $N_0 < 1$, and has an order of smallness greater than the first at zero; finally, the spacing between $\Pi^n(u; \varepsilon, \mu)$ and this curve decreases at the rate of a geometric progression as $n \to \infty$. The function $v_0(\eta; \varepsilon, \mu)$ still possesses a certain smoothness in η and ε [7]. This permits establishment of the monotonicity of the operator (5.3) in the elements $l(\varepsilon, \mu)$ and then the use of the scheme and analog to the Lemma 2.19 described in [6] to complete the proof.

The proof of Theorem 3 follows from the monotonicity of the operator (5.3) in the elements $l(\varepsilon, \mu)$ and from what has been elucidated in [7]. The proofs of Theorems 4 and 5 are carried out approximately in the same manner as in [6].

6. A supplement. Let $b_2(0) \neq 0$. We consider an arbitrary solution $x(t)$ of (1.1) with initial conditions from the sphere $S(r_0)$. Then for values of t at which the trajectory of the solution $x(t)$ remains in $S(r_0)$, the formula

$$x(t) \approx \eta(t)\, E_1\left(\frac{t + c_0}{1 + c_2(\mu)\, a_1^2(\mu)\, \varepsilon}\right) \tag{6.1}$$

analogous to (4.3), is valid. Here $\eta(t)$ is a solution of the equation

$$\eta^{\cdot} = \varepsilon \tau_0' \eta - b_2(\mu) \tau_0' \eta^3, \qquad \eta(0) = \eta_0 \geqslant 0$$

and the constants c_0 and η_0 depend on the initial conditions of $x(t)$. There results from Sect. 5 that terms which are either proportional to ε or damp exponentially with the exponent independently of ε, are discarded in the right side of (6.1).

7. Conclusion. The method proposed is applicable to nonconservative problems of elastic stability theory in which the loss of stability of the equilibrium state occurs by oscillations. To confirm the corresponding fact, the spectrum of the linearized problem must be analyzed. Afterwards, a certain quantity of stationary inhomogeneous equations must be solved. This latter can be done by relying on the Galerkin method, for instance. This program of operations is described in detail in [12] in which the nonlinear flutter of an essentially two-dimensional panel is computed numerically and compared with experimental results published by American authors. The discrepancy from experiment did not exceed 8%.

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